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Improving High School Mathematics Teaching

Students must be skillful computers in order to work effectively with ideas.

LET us drop in on a ninth grade algebra class in the traditional mathematics curriculum. The teacher introduces the lesson by telling the class:

Today we shall learn how to add terms, and we shall also learn how to tell when they cannot be added. Who remembers what a term is?

A student replies:

A term is an indicated product of numbers and letters.

The teacher nods his head in agreement:

Yes, we shall learn how to do problems like $2a + 3a$, $4b + 7b$, $8x + 3y$, etc. How do we add $2a + 3a$?

Then, since the question is rhetorical and since the student needs to "understand" the problem, he continues:

Suppose Mary has two apples and Bill has three apples. How many apples do they have altogether? . . . Yes, they have five apples altogether. So, $2a + 3a = 5a$. Just add the numbers and write the letter next to the answer. Who knows what $4x + 7x$ is ? . . . Good, yes, $4x + 7x = 11x$. Now, turn to your textbook, and do the first 10 exercises on page 34.

These exercises are just like the ones discussed by the teacher, and the class finishes them very rapidly. Next, the teacher poses the question:

What is $2a + 3b$? Suppose John has two apples and Susan has three bananas? Can you add apples and bananas? (Rhetorical question.) Of course not. Therefore, you cannot add $2a + 3b$. You can only indicate their sum. Thus, we see that when the terms are like, you can add them. When terms are not like, you can't add them. Now do exercises 11 through 20 on page 34.

There are at least two things wrong with the kind of teaching illustrated above. For one thing, the "explanations" offered by the teacher have not the remotest connection with mathematics. Secondly, the students are passive, and are encouraged to acquire a skill through imitation alone. The University of Illinois' project for the improvement of the teaching of secondary school mathematics seeks to bring mathematics into the teaching of mathematics, and to encourage the learner to discover as much of the subject as time and circumstances will permit.

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Using Abstractions

The subject matter of mathematics consists of abstractions. Most everyone agrees with this statement. Among those who do is the teacher mentioned in the foregoing illustration. He also believes that learning proceeds most efficiently if you go from the concrete to the abstract. So, he "helps" his students by going from apples and bananas to 'a' and 'b', because he thinks that 'a' and 'b' are mathematical abstractions. (He also believes that equations are abstractions, and so in a lesson on equations he teaches students to solve equations by talking about scales or balances.) This distorted view of what mathematical abstractions are is a major block to effective teaching.

A mathematical abstraction is not a symbol or a mark which someone makes on paper or on a chalkboard. A mathematical abstraction is an entity which has no physical existence. A person becomes aware of a given abstraction after dealing with several instances of it. For example, justice is an abstraction, and as a child observes instances of justice and instances of injustice, he gradually reaches the point where he can classify a certain act as being an instance of justice or an instance of injustice; that is, as being a just act or an unjust act. Ideally, the child also learns the word 'justice' sometime after he has learned to recognize instances of the abstraction justice. He views the word 'justice' as a convenient label to use in talking about justice; he does not say that the word itself is the abstraction.

The same point of view holds for mathematical abstractions. A child becomes aware of the abstraction 3 by playing with objects. He notes instances of 3, such as a set of 3 blocks, a set consisting of a sandwich, a cookie, and an apple, and a set of 3 cars. After considerable experience, he becomes aware of a common property of these sets (or, of the collection of all such sets). At this point, he can tell you whether a given set is an instance of 3. The teacher can use such behavioral evidence to tell when the child has become aware of the abstraction. This is the time to introduce the child to the symbol '3' (or 'three' or 'trois' or 'drei'). The symbol '3' is not the abstraction. The abstraction is what the symbol stands for, to wit, the number 3. The number 3 is not the set of 3 apples nor the set of 3 food items, nor the set of 3 cars. Each of these sets is a "concrete" instance of the abstraction. All of this becomes perfectly evident when we observe normal children acquiring skill in the use of common nouns. A common noun, of course, is a symbol which stands for an abstraction. When a child no longer uses the word 'doggie' when he sees a cat, his parents feel that he is becoming aware of the abstraction dog.

Even though the proper subject matter of mathematics consists of abstractions, the teaching of mathematics is largely characterized by attempts to teach children efficient methods for manipulating symbols without giving children sufficient experience to become aware of the abstractions denoted by the symbols. Unfortunately, we pedagogues are ingenious enough to invent all kinds of mnemonic devices which enable students to remember mechanical procedures for manipulating symbols. We use mnemonic devices in the form of words like 'borrow', 'carry', and 'cancel', and in the form of rules and admonitions like 'invert and multiply', 'you can add like fractions but not unlike fractions', and 'like terms can be combined but unlike ones cannot'. Some students can be brought to a high level of mastery of skills through devices of this kind. They perform well on tests which require them to demonstrate the skills in familiar settings. The result is different when we hide the familiar skill in a new situation, or when we test students after a summer vacation. In the former case, the student must be able to classify the new situation in terms of abstractions before he can manipulate.[1] In the latter case, unless the student has a really good memory (or unless he is one of the rare ones who is able to see connections between the abstractions and the mechanical procedures), he tells you that he isn't sure whether he should get a common denominator in order to multiply, or that he can't remember which fraction to invert, or that he's not sure whether $2a + 3b$ is $5ab$ or $6(a + b)$.

Now, it may very well be the case that the learning of mathematics *should* consist of lots of memorizing, with the prizes going to those who have the best memories and to those who have talent enough to enable them to put the abstractions into some sort of consistent structure and, when memory fails, to derive the mechanical procedures from this structure. Perhaps the current shortage of people who feel comfortable with mathematics can be best eliminated just by putting more emphasis on the memory work. If so, then most of the current mathematics curriculum reform groups are moving in the wrong direction. In particular, if the only way to get results is to give lots of practice in using mnemonic devices and not waste time on abstractions (that is, not waste time on mathematics), the program of the University of Illinois project for the improvement of mathematics teaching is destined to fail, and the interesting results already observed must be illusory.

[1] Dean Brownell tells the story of the second grader who floundered when his teacher gave him the problem: If 5 birds are sitting on a telephone pole and 3 fly away, how many are left? After listening to wild guesses such as 12, 6, and 5, the teacher said in desperation, "Look, 5 take away 3 is how much?" "Oh," said the student, "5 take away 3 is 2. Why didn't you ask me that in the beginning?"

Improving Materials

The University of Illinois Committee on School Mathematics (UICSM) was appointed in December 1951 by the deans of Education and Engineering and by the head of the Mathematics Department. The four-member committee was asked to investigate procedures for implementing the recommendations made by an earlier committee[2] for improving the competence of beginning engineering students to the point where they would be able to study calculus in the freshman year. It was quickly decided that a realistic proposal for improvement would have to include classroom-tested instructional materials.

In September 1953 a ninth grade course was tested in two public high schools in Illinois (Barrington and Blue Island) and in the University of Illinois High School. In the ensuing years we produced instructional materials for higher grades, we continually revised our earlier efforts, we increased the number of schools experimenting with the program, and we found it necessary to train the teachers who wanted to work with us.

In the current school year (1959-60) over 80 schools throughout the country are trying the UICSM program, our staff numbers 17, and the *eighth* version of

[2] University of Illinois College of Engineering. *Mathematical Needs of Prospective Students.* First edition, 1951; revised edition, 1959. Urbana: Office of Field Services, College of Education, 1959. 28 p.

the instructional materials for the beginning course has been published and released to the profession.[3] This expansion has been made possible financially by two 3-year grants from the Carnegie Corporation of New York amounting to a total of over a half-million dollars. The University of Illinois carried the complete financial burden of the project from its inception in 1951 to the time of the first grant in July 1956. The project was a "going concern" by the time its merit was recognized by the foundation, and the fact that the University had invested considerable money and staff time (and is still doing so) probably played an important part in attracting the support of an outside agency.

Foundation support has enabled us to use the full-time efforts of research mathematicians and high school teachers in developing instructional materials which are in harmony with our point of view regarding mathematics and its teaching. Most textbooks designed for the traditional curriculum place too much emphasis on recipes for manipulating symbols and give too little attention to the substance of mathematics itself. Although many textbooks do attempt to take care of "the why as well as the how," these attempts are very much like those of our teacher at the beginning of the article. They are primarily concerned with mnemonics.

As mentioned earlier, we believe that a student needs to become aware of mathematical abstractions before he begins manipulating the symbols which denote them. When he reaches the symbol-manipulating stage, he first manipulates according to what he considers "common sense." Quite frequently, common sense does not dictate the most efficient procedure, so the student then begins the search for short cuts, for algorithms, for mnemonic devices. Many students succeed in inventing such procedures and take considerable pride not only in the fact that their rules work, but also that they can prove that the rules must work. For in putting the major emphasis on abstractions rather than on symbols, the teacher enables the student to discover certain basic relations among the abstractions. The student notes, for example, that a first number multiplied by a second yields the same result as the second multiplied by the first. This property of numbers and multiplication is known as the commutative principle of multiplication.

Again, the student becomes aware of a principle to the effect that if you multiply each of a first number and a second by a third and add the products, you get the same result by adding the second to the first and multiplying the sum by the third. He gains this awareness by looking for easy ways to solve problems like:

$$7 \times 8 + 3 \times 8 = ?$$
$$7 \times 92 + 3 \times 92 = ?$$
$$63 \times 49 + 37 \times 49 = ?$$

By the time he reaches the problem of simplifying an expression like '2a + 3a', he jumps immediately to '5a', and tells you, if you ask, that '2a + 3a = 5a' is a consequence of the principle mentioned above and the fact that 2 + 3 is 5. If you ask him *how* he gets '5a', he will tell you his short cut. But, there is no need to ask the how-question. Any fool can plainly see *how* he does it! The important question is the one which asks for an explanation of *why* the short cut works. And the only mathematical answer is the

[3] UICSM. *High School Mathematics*. Units 1, 2, 3, & 4. Teachers' edition. Urbana: University of Illinois Press, 1959. 1432 p. (Looseleaf, photo-offset.)

one which refers to properties of mathematical abstractions.

The UICSM instructional materials are full of exploratory exercises designed to build awareness of abstractions. When we hope that a student will invent a short cut as a result of his work on, say, page 18, we do not give the game away by stating the short cut on page 19, or page 20, or anywhere in the textbook, for that matter. Also, we don't "beat the drums" for mathematics by telling the student that the subject will be important to him when he grows up. If we cannot make it important to him while he is studying it, we cannot justify making him study it. But, it is not difficult to make mathematics important to young people. When a child invents a rule which saves him labor and which he can prove is correct, no one needs to convince him that the rule is important. He has firsthand experience to support the thesis that mathematics is a living subject, and not just something embalmed in textbooks.

Disseminating Ideas

Foundation support has also made it possible for us to disseminate these ideas to teachers. We have developed special editions for teachers which accompany the instructional materials for students. These teachers' editions are really compendia of pedagogy and mathematics. In them we can defend the teaching approach used in the students' materials, and we can emphasize again and again the distinctions between mathematical abstractions and the corresponding symbols and the need for exploratory work to precede the symbolization. We can show the teacher how the mathematical structure of which the student is gradually becoming aware is really part of a much larger structure. We can show how a foundation is being laid for more advanced ideas. Finally, we can give very specific assistance in the form of accounts of successful alternative procedures used by other teachers in the preceding years of experimentation.

Another type of dissemination which the foundation supports is our Teacher Associateship program. Each year we invite five or six teachers to take a year's leave of absence from their present positions and spend it on our staff. (We contribute a fixed amount to the salary, and the sponsoring school system or college pays the rest.) Teacher Associates are given very light teaching assignments and spend most of their time in studying the content of our curriculum and in helping us work with teachers in cooperating schools. When a Teacher Associate returns to his own institution, he may want to revise the methods courses he usually teaches. If he is a teacher or a supervisor in a secondary school system, he may want to conduct in-service courses.

It should be clear that the UICSM does not sponsor a crash program. (In fact, we have been accused of being hairsplitters and perfectionists.) Nor do we advocate the wholesale substitution of so-called modern mathematics for the content of the traditional program. To be sure, the traditional program has developed certain excesses, particularly in the areas of complicated computation and manipulation, and we are in accord with other groups who advocate deemphasizing these topics. Also, we have brought into the curriculum a few ideas which mathematicians first became aware of in the latter half of the nineteenth century, and in the early years of the present century. The ideas are commonly included

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Parlons Français

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one of three years of televised courses for teachers and children which are ultimately to be filmed in a final form after appropriate revisions of program content and presentation. This work is a very exciting prospect for all involved in the project, for the programs will thus be available on a national basis. It is planned to have a whole package for a year's course of films, guides and recordings. We are also hoping that material filmed in France will add authenticity not otherwise attainable.

In conclusion, a word about the research which will be made on the program. Since this project is experimenting in somewhat new fields, the federal government, in considering the original research proposal entailing funds under Title VII of the National Defense Education Act, requested that the research design be made even more thorough and extensive. Forty-eight fourth grade classes have been selected from ten suburban school systems in the Boston metropolitan area in communities which are not bilingual and with teachers selected on the basis of experience and interest in the project. The research design proposes to compare the effectiveness of several combinations of televised teacher-training and direct instruction in oral French upon the achievement in listening and speaking skills of fourth grade children. Tests are being administered to teachers at the beginning and the end of the school year, while the children will be tested only after the programs are over, on the assumption that a minimum percentage of children are already bilingual. In view of the critical need for teachers of foreign languages at the elementary level, the results of this research should provide some definite information on in-service teacher training and on the success of the combination of expert TV teacher and elementary classroom teacher.

Mathematics Teaching

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in the branches of mathematics called 'set theory' and 'foundations'. We use them because they enable students to see more readily the connections between manipulative procedures such as equation-solving and graphing and the structure of the number system. But, whether an abstraction is a twentieth-century one or one known to the Pythagoreans, our working rule is that students need to develop an awareness of it, symbolize it, and then invent mechanical procedures for manipulating the symbol.

In ending this discussion I must point out that our attempt to get students to see that mathematics is an intellectual subject should not be misconstrued as an attempt to have students deal only with ideas and to eliminate manipulative drill. Drill is absolutely essential in developing skills, and students must be skillful computers in order to work effectively with ideas.