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Preparing Elementary Teachers in Mathematics

Involved are creativity, logic and understanding.

A FIFTH grader who answered $7+?=5$ with "two in the hole," and symbolized it with ②, had invented a wealth of possibilities. Most fifth graders, and many teachers, would have replied, "There is no number which can be added to 7, giving 5." This boy created something new, and he was excited about it. He wanted to add these new "in-the-hole numbers." After that, he wanted to subtract them. Recalling that $10-8$ is another name for the number 2 because $8+2=10$, he concluded that $10-⑧$ must be another name for the number 18, because $⑧+18=10$.

The significance of this incident lies not in the child's early use of "in-the-hole numbers," but in the tremendous importance of his creativity and the understanding and logic demonstrated in his solution.

At one time, the arithmetic teacher was expected to turn out about 30 little computers each year. Today, machines are faster, more accurate computers than man will ever be. While it is true the teacher must help children develop computation skills, the creativity and reasoning ability of humans is considered far more valuable treasure than their ability to compute.

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Discovery and Proof

Inherent in the new programs in mathematics is a method of teaching which encourages student discovery, creativity, and proof. This method demands more of the teacher, and necessitates different textbooks. A teacher must know more mathematics to lead students to discovery than was needed simply to justify a rule. Textbooks which provide a poorly worded rule in a red box, followed by an example and 25 exercises requiring only that the reader follow directions, are not compatible with the new look in mathematics.

In many teacher training programs, the prospective teacher has been placed in a class reviewing arithmetic. This is what he did in grade 7, grade 8, and grade 9. By the time the student reaches college, he has become so familiar with the rules, "cross multiply," "change the sign and add," "invert and multiply," that he can and does misuse them without even thinking. The short course in symbol-juggling algebra or the review of arithmetic seems to be designed to kill or cure, with little real possibility of a cure.

The prospective teacher needs to

spend time with mathematics. He needs time to become interested, to overcome fears, to learn some mathematics, and to let understanding come into being.

There are fundamental principles of mathematics which, with proper definition, can be used to develop all the rules of arithmetic. College students have become so familiar with our number system that these principles are not apparent to them. These students, in training for teaching, need some of the new mathematics. They need the precise definitions, the "set" terminology when it clarifies concepts being discussed, and more symbolism when it aids the thinking. They need the opportunity to enjoy student discovery, for it is likely that they will teach as they have been taught.

One approach with an emphasis on the fundamental principles and mathematical structure requires that prospective teachers work with unfamiliar systems setting the stage for a fresh look at mathematics. They make use of an addition table such as:

⊕	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

In this approach, the student must depend on the table, for this obviously is not his familiar addition. He finds that $2 \oplus 3 = 0$, $3 \oplus 4 = 2$, and that $4 \oplus 3 = 2$. First, the student will notice differences between our number system and this new one. Indeed, he may be surprised to discover any similarities. In playing with this new structure, the student discovers:

1. Any first "number" added to any second "number" gives the same result as if the second "number" were added to the first "number."

2. Any first "number" added to the sum of the second and third "numbers" gives the same result as the sum of the first and second "numbers" added to the third "number."

3. There is a "number" 0 such that 0 added to any "number" gives that "number."

4. Adding any two of the "numbers" gives a unique "number" in the table.

He will be interested to find that these principles are also properties of our number system. He will later use them and their counterparts for multiplication to prove the rules of arithmetic.

Referring again to the table, the student is presented with a subtraction problem similar to that of the fifth grader. When attempting to find a simpler name for $2 \ominus 4$ he finds that "take away" is not a fitting definition for subtraction. The student must conclude that $2 \ominus 4 = 3$ if and only if $4 \oplus 3 = 2$.

By defining multiplication as repeated addition, the student can complete a multiplication table for this system. Examination of the table will lead to the discovery of the counterparts to the four principles already listed. He has now identified at least eight of the fundamental properties of numbers, but he has done so on a new frontier. Definitions which were once useful only in examinations, now become useful in solving problems.

After this brief introduction, the student is ready for a development of the real numbers. He should be well enough acquainted with the principles of numbers to question any text which states: "3×0 is a multiplication fact, because you can have a score of zero three times, but 0×3 is not a multiplication fact, because you cannot have a score of three zero times."

As a teacher, he should be able to distinguish between numbers and numerals. He should see that in order to perform all subtraction problems, it is necessary to invent the negative numbers. He should see that to perform all division, except division by zero, it is necessary to invent still more numbers. He should recognize the significance of a one-to-one correspondence. He finds that number is an abstract concept. He is not adding two cows and three cows, rather he is adding the number 2 and the number 3, or he is finding a simpler name for $2+3$.

A great deal of mathematics time will be spent on the set of rational numbers. This is the set of numbers which can be represented as fractions. In addition to learning something about proof and something about rational numbers, the student should see continuity in mathematics. He has already learned one of the fundamental principles: there exists a number, 1, such that for any number a, $a \times 1=a$. The student also knows that a number has many names, in particular that $7-6$, $\frac{1}{2}+\frac{1}{2}$, $3(\frac{1}{3})$, $\frac{5}{5}$, $\frac{17}{17}$ are all names for the number one.

The teacher who emphasizes the fundamental principles of numbers may assign problems such as $3\overline{)12}$, $6\overline{)24}$, $3\times 19\overline{)12\times 19}$. These lead to a generalization that if $n \neq$ (is not) 0, $3n\overline{)12n} = 3\overline{)12}$. During the study of rational numbers, $\frac{3}{5} \div \frac{2}{3}$ may be written $\frac{2}{3}\overline{)\frac{3}{5}}$ which names the same number as $\frac{3}{2}n\overline{)\frac{3}{5}n}$ if n is not zero. Even though he may not know what number is named by $\frac{2}{3}\overline{)\frac{3}{5}}$, the student will be able to convert this to an equivalent expression for which he will know the number named: $\frac{2}{3}\times 15\overline{)\frac{3}{5}\times 15}$ or $\frac{2}{3} \times \frac{3}{2}\overline{)\frac{3}{5} \times \frac{3}{2}}$ or even $\frac{2}{3} \times 3\overline{)\frac{3}{5} \times 3}$. From his past experience and from use of the principle of one, he knows that the number named by any one of these is the number named by each of them.

Still later, the teacher uses the same principle to teach $.25\overline{)7.5}=30$. The usual method is to "move the decimal points." In other words, we admit that we cannot solve $.25\overline{)7.5}$, so we solve $25\overline{)750}$ instead. What reason is there for assuming that these two problems name the same number? Once again $.25\overline{)7.5}=.25n\overline{)7.5n}$ from the principle of one. This means that $.25\overline{)7.5} = .25 \times 100\overline{)7.5 \times 100} = .25\times 4\overline{)7.5\times 4}=.25\times 3\overline{)7.5\times 3}$ and in this case $.25\times 4\overline{)7.5\times 4}$ is the simplest problem to solve.

The very same principles are used later when the student rationalizes denominators, numerators, or performs the indicated division with complex numbers.

A program for the preparation of teachers of mathematics is not one that can be completed in one term, one semester, or even one year. The content course in mathematics, at least nine quarter hours, must be supplemented by a course relating this content to the mathematics of the elementary school, and one which emphasizes methods and materials consistent with this mathematics and the psychology of learning.