thinking aloud as they solve problems like the one shown earlier. Then they switch to physics or engineering problems but continue to work in pairs aloud. In contrast, various high schools and colleges will have weaker students spend the entire semester solving such problems. They focus entirely on analytical skills without involving specific content from physics, mathematics, or other subjects.

Project SOAR (Stress On Analytical Reasoning), a five-week pre-first year program at Xavier University in New Orleans, included Piagetian-based science laboratories in the morning, and vocabulary expansion, problem solving, and reading exercises using the thinking aloud format in the afternoon. These gains were reported by SOAR's director:

> The 34 students who initially scored below grade 12 on the comprehension section of the Nelson-Denny Reading Exam showed an average improvement of 1.4 years.
>
> The 43 students who scored below grade 12 on the vocabulary section of the Nelson-Denny gained an average of 1.8 years.
>
> The 21 students who scored less than 70 on the PSAT (equivalent to 700 on the SAT) gained an average of 11.4 points, and the entire group (113 students) gained 7.3 points. These were equivalent to gains of 114 and 73, respectively, on the SAT (Carmichael, 1979, p. 2).

In 1960 a front-page article in *The New York Times* reported an average IQ increase of 10.5 points for the 280 first year students at Whittier College who took a two semester course designed by English professor Albert Upton. The pedagogy of Upton's program was very similar to that of Bloom and Broder. Classes met four times a week; once for lecture and three times in small groups with tutors for problem solving with verbal comprehension and reasoning problems.

Upton's approach has been incorporated into a remedial language arts program called THINK, spanning reading levels one-12, and a parallel math program called Intuitive Mathematics.[2] The problem solving/discussion format used by both programs is proving very effective. In studies by June Gabler (1977), superintendent of the Woodhaven school district in Michigan, THINK produced larger gains on the Nelson-Denny than those produced by stand-

[2] Both programs are available from ISI Think, Inc., 300 Broad St., Stamford, CT 06901.

Problem Solving in Mathematics: National Assessment Results*

An important goal of the mathematics curriculum is to teach students to apply the mathematics they are learning to new or unfamiliar situations. Recently released results from the second mathematics assessment of the National Assessment of Educational Progress (NAEP) indicate that although students are learning many basic algorithmic or computational skills, they have difficulty applying these skills to solve even simple nonroutine problems.

The NAEP mathematics assessment results are based on the performance of a representative sample of over 70,000 9-, 13-, and 17-year olds who took a carefully developed set of about 500 exercises that assessed important objectives of the mathematics curriculum. The results of this assessment represent the best available measure of American students' mathematical achievement.

Students were generally successful in solving simple textbook problems that could be solved by applying a single operation to the numbers given in the problem. However, any exercise that required students to do more than decide whether to add, subtract, multiply, or divide caused considerable difficulty.

At all three age levels, students would frequently attempt to apply a single mathematical operation to whatever numbers were given in a problem rather than analyzing the problem to decide how to solve it. For example, only about 10 percent of the 9-year-olds and 30 percent of the 13-year-olds correctly solved the following problem:

> Mr. Jones put a wire fence all the way around his rectangular garden. The garden is 10 feet long and 6 feet wide. How many feet of fencing did he use?

Almost 60 percent of the 9-year-olds and 40 percent of the 13-year-olds simply added the 6 and the 10.

An important aspect of problem solving is identifying which facts are relevant to a given problem. The following item, which includes extraneous data, illustrates the difficulty students had in analyzing a problem.

> One rabbit eats 2 pounds of food each week. There are 52 weeks in a year. How much food will 5 rabbits eat in a week.

Almost a fourth of the 13-year-olds attempted to incorporate all three numbers given in the problem into their calculation.

When students could identify the appropriate operation, they frequently had difficulty relating the result of their calculation to the given problem. The following exercise involves a simple application of

* This material is based upon work supported by the National Science Foundation under Grant No. SED-7920086. Any opinions, findings, and conclusions or recommendations expressed in this publication are those of the authors and do not necessarily reflect the views of the National Science Foundation.

ard reading instruction for below average, average, and above average students in both middle and high schools. Also, Intuitive Mathematics resulted in average gains on the Metropolitan Achievement Test which were three times those from standard instruction for high school students with histories of difficulties in mathematics (Gabler, 1977).

Taking a somewhat different tack, Jill Larkin (1977) compared graduate students with outstanding physicists at Berkeley on the way they solved certain physics problems. By having them think aloud she found that the graduate students tended to be "formula bound" while the physicists used mental pictures and other analytical devices to understand situations before starting their computations. Booklets were then developed which successfully taught the analytical strategies to introductory physics classes. In this case the training did not involve small discussion groups; students worked alone, but the booklets did require active responses from the students as they practiced the mental activities reported by the outstanding physicists. Research indicates that small group discussion is useful at all ability levels but is most crucial for the students at the lower levels.

The emphasis on thinking skills is reaching the highest educational settings, as illustrated in these excerpts from a description of a program at McMaster University School of Medicine, written by the director, Howard Barrows (1979).

> The problem solving skills of physicians are central to their ability to apply health care effectively and efficiently. McMaster University School of Medicine has chosen problem based, self directed learning in small groups as the principal format for learning undergraduate medicine. One of the objectives of problem based learning is to help students develop appropriate cognitive skills in medical problem solving. Therefore, a group of us in the School of Medicine completed a five-year study of the problem solving process of the physician.
>
> The next stage of our work was to design appropriate learning units to develop this problem solving process in postgraduate and continuing medical educational settings.
>
> At the present time we are formulating an overall sequence for the development of the problem solving process of the medical student from small group learning around the simulated patient, through individual tasks with simulated patients, ending with structured clinical situations with real patients under pressure to finely tune the problem solving process.

long division. But the answer to the problem is the remainder to the division calculation, not the quotient.

> A man has 1310 baseballs to pack in boxes which hold 24 baseballs each. How many baseballs will be left over after the man has filled as many boxes as he can?

Fewer than 30 percent of the 13-year-olds correctly answered this problem. Over 20 percent gave the quotient as their answer.

Students' mechanical application of computational algorithms often resulted in unreasonable answers that they should have recognized if they had thought about the problem. For example, consider the multiple choice estimation exercise in Figure 1. Many of the students at each age level simply added the numbers in the numerator or denominator. The magnitude of the result is completely unreasonable if one understands what it means to add two fractions less than one. However, rather than estimating the sum, many students appear to have attempted to find some calculation involving the numbers given in the exercise with no concern for the reasonableness of the result. Performance was over 15 percentage points better on a direct computation exercise in which students were asked to calculate the sum of the two fractions.

Figure 1. Sample Estimation Problem

ESTIMATE the answer to $\frac{12}{13} + \frac{7}{8}$.

	Response	Percent Responding Age 13	Age 17
0	1	7	8
0	2	24	37
0	19	28	21
0	21	26	15
0	I don't know	14	16

The results for the exercises described here as well as many other exercises on the assessment strongly suggest that students have become accustomed to mechanically applying mathematical algorithms to problems without analyzing the problems or thinking about the reasonableness of their answers. In recent years, a great deal of attention has focused on the teaching of basic skills, which are frequently equated with routine computational skills. Mathematics skills, however, have little value if they cannot be applied to new or unfamiliar situations.

The ability to analyze a problem situation is as important and as basic a skill as the ability to compute the answer. The assessment results suggest that problem solving is the basic skill most in need of attention in the mathematics curriculum.

Thomas P. Carpenter, Mary Kay Corbitt, Henry Kepner, Mary Montgomery Lindquist, Robert E. Reys.