

# Evaluating Problem Solving in Mathematics

*Effective assessment of problem solving in math requires more than a look at the answers students give. Teachers need to analyze their processes and get students to communicate their thinking.*

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In its *Curriculum and Evaluation Standards for School Mathematics*, the National Council of Teachers of Mathematics expanded the goals it developed in 1980 for promoting problem solving as a curricular focus (NCTM 1989). The first three standards — Mathematics as Problem Solving, Mathematics as Reasoning, and Mathematics as Communication — show a shift from emphasis on rules and routine problem solving dominated by teacher talk and passive learning, to active student participation, in which reasoning and communicating are stressed.

These efforts are admirable, but they create new challenges, especially in assessment of these higher-level skills. Problem solving requires considerable thinking, but even when students are able, they are not inclined to communicate their thinking. Without such communication, how can we reliably assess students' efforts to solve problems? Before discussing how to improve communication and assessment, it is useful to clarify the notion of a problem and problem solving.

## The Nature of Problems and Problem Solving

Problem solving is the process of

confronting a novel situation, formulating connections between given facts, identifying the goal, and exploring possible strategies for reaching the goal. A problem, then, is a situation in which the individual initially does not know any algorithm or procedure that will guarantee solution of the problem, but the individual desires to solve it.

Success in problem solving depends upon metacognitive processes, as described by Garofalo and Lester (1985). The following list summarizes the typical sequence of actions for successful problem solving:

1. Obtain an appropriate representation of the problem situation.
2. Consider potentially appropriate strategies.
3. Select and implement a promising solution strategy.
4. Monitor the implementation with respect to problem conditions and goals.
5. Obtain and communicate the desired goals.
6. Evaluate the adequacy and reasonableness of the solution.
7. If the solution is judged faulty or inadequate, refine the problem representation and proceed with a new strategy or search for procedural or conceptual errors.

These metacognitive processes are difficult to assess, but assessment can be expedited by creating problem situations that facilitate students' communication of their thinking.

## Difficulties in Assessment of Problem-Solving Performance

The difficulty of assessing complex processes necessary for solving problems is exacerbated by the failure of students to communicate clearly what they have done or what they are thinking. Students are prone to make calculations without explanations, and calculations alone often fail to reveal sufficiently the nature of the solver's work and thinking. It is not enough to

FIGURE 1  
ANALYTIC SCALE FOR  
PROBLEM SOLVING

### Understanding the problem

- 0 - No attempt
- 1 - Completely misinterprets the problem
- 2 - Misinterprets major part of the problem
- 3 - Misinterprets minor part of the problem
- 4 - Complete understanding of the problem

### Solving the problem

- 0 - No attempt
- 1 - Totally inappropriate plan
- 2 - Partially correct procedure but with major fault
- 3 - Substantially correct procedure with minor omission or procedural error
- 4 - A plan that could lead to a correct solution with no arithmetic errors

### Answering the problem

- 0 - No answer or wrong answer based upon an inappropriate plan
- 1 - Copying error; computational error; partial answer for problem with multiple answers; no answer statement; answer labeled incorrectly
- 2 - Correct solution

check for right and wrong answers or to use multiple-choice formats for assessment of problem solving. As Silver and Kilpatrick (1988) state:

A reliance solely on the sleek efficiency of multiple-choice (and other short answer) formats will severely hinder efforts to help students develop a reflective and interrogatory stance toward their learning.

If we can devise methods for eliciting better communication of students' thinking, we can perform more effective assessment. Such assessment measures the quality of students' thinking. This information can help teachers design and implement instruction to promote greater success in problem solving and can help administrators evaluate programs and curriculums.

### Assessment of Solved Problems

The most natural and common method for assessing performance in problem solving is to obtain general impressions about the quality of a solution while scanning students' work. These general impressions are strongly influenced by the "proximity of correct-

ness" of the answer. As a result, good solutions with minor errors due to carelessness that alter the answer dramatically can receive undeservedly low scores. Scales are available that focus more attention on solution procedures, enabling teachers to obtain fairer and more reliable scores. For example, Charles, Lester, and O'Daffer (1987) devised a scale that assigns separate scores to each of three stages in problem solving: understanding the problem, solving the problem, and answering the question. Figure 1 shows a modification of their scale, with increased emphasis given to the understanding and solving stages (Wilson 1991).

The Charles, Lester, and O'Daffer scale and its modified forms are easy to use. An advantage of such a scale is that a teacher may focus on only one of the stages. For example, a teacher who is emphasizing strategy selection and implementation can assess each student's solving procedure irrespective of the answer.

The California Assessment Program (Pandey 1990) includes comprehensive descriptions of various levels of performance for specific problems. This is

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appropriate for large-scale assessment programs. However, the classroom teacher has little time to construct scales for individual problems. Teachers need assessment procedures and scales that they can modify or use intact for a wide range of problems.

### Categorizing Responses to Problems

Scales for assessment of problem solving can be designed without creating an evaluative threat to students. Such a system of scales was constructed for use in the 1990 British Columbia assessment of problem solving (Szetela 1991). Instead of scoring the solutions only, teachers analyze the responses to problems on the basis of four categories: answers, answer statements, strategy selection, and strategy implementation (see fig. 2). Teachers can use a single category to determine how well their students are addressing a particular aspect of solving problems. One focus might be on strategies used. Another might be directed toward answer statements. Incomplete statements that fail to include the units taught or important contextual information may serve as focal points for teachers in their subsequent instructional activities.

FIGURE 2

#### CATEGORIES OF RESPONSES IN SOLUTIONS TO PROBLEMS

Answer	Strategy Selected	Implementation
1. Blank	1. Number sentence	1. No work shown
2. Undetermined	2. Select operations and calculate	2. Identifies data only
3. Incorrect	3. Algebraic	3. Problem misinterpreted
4. Correct	4. Non-systematic list	4. Strategy not clear
	5. Systematic list	5. Strategy initiated (table, graph, list) but incomplete or poorly implemented
	6. Guess and test	6. Conditions or possibilities overlooked
	7. Draw diagram	7. Multiple secondary errors
	8. Look for pattern	8. A single secondary error
	9. Logical reasoning	9. Appropriate and complete
	10. Use simpler case	
	11. Work backwards	
	12. Undetermined	
Statement		
1. No statement		
2. No context		
3. No units		
4. None required		
5. Complete		

## Students are prone to make calculations without explanations, and calculations alone often fail to reveal sufficiently the nature of the solver's work and thinking.

### Promoting Greater Communication

To further enhance assessment, we need to devise problem situations and questions that encourage and motivate students to communicate and explain their thinking. Figure 3 shows one way to do this. An already solved problem with a significant error, combined with a set of relevant questions about the solution, facilitates communication. As with an unsolved problem, students must form a suitable representation of the problem. Instead of solving the problem themselves, however, they analyze the given solution. Finally, they reveal their thinking by answering the pertinent questions. Answers to these questions can provide more comprehensive insights about the student's thinking in problem situations than more typical problem formats, in which students may have various levels of success but fail to reveal their thinking.

Assessment of responses to the questions accompanying the already solved problem can be done in less time than it normally takes for teachers to plod through the usual

wide range of solution procedures for a given problem. The main goal of the example in Figure 3 is to determine whether or not a student understands a problem situation well enough to recognize the incongruity of the given answer despite excellent implementation of a good plan, with the problem solver running awry only in the careless writing of the answer statement. Teachers can provide continuing experiences for students to critically analyze solutions and communicate their observations and responses to relevant questions. Such practice can help students engage in reasoning, evaluating, and communicating, and can enable teachers to assess these problem-solving processes more effectively.

Other forms of problems with questions to stimulate thinking and written communication include the following:

- Present a problem with all the facts and conditions, but have the students write an appropriate question, solve the completed problem, and write their perceptions about the adequacy of the solution.
- Present a problem and a partial solution. Have students complete the solution.
- Present a problem with facts unrelated to the question. Have students comment about the quality of the problem or revise the problem to remove the incongruity.
- Have students explain

how they would solve a problem using only words, then solve the problem and construct a similar problem.

- After students solve a problem, have them write a new problem with a different context but preserving the original problem structure.
- Present a problem without numerals. Have students supply appropriate numerals, estimate answers, and solve the problem.

Teachers can assess the quality of each response by using a scale such as the following:

FIGURE 3

### Example of Problem that Asks Students to Communicate Thinking

A bowl contains 10 pieces of fruit (apples and oranges). Apples cost 5 cents each and oranges cost 10 cents each. All together the fruit is worth 70 cents.

We want to find how many apples are in the bowl.

Kelly tried to solve the problem this way.

$$10 \times 5 = 50$$

$$2 \times 10 = 20$$

$$70$$

$$8 \times 5 = 40$$

$$3 \times 10 = 30$$

$$4 \times 10 = 40$$

$$6 \times 5 = 30$$

*I here was  
30 apples in the bowl.*

Try to follow Kelly's work and solution. Then answer the questions.

1. Is Kelly's way of solving the problem a good one?

*Yes*

Tell why you think it is or is not a good way.

*because it will tell you the possible solution which is what you want but she didn't read carefully*

2. Did Kelly get the right answer?

*No*

Explain why she did or did not.

*because there are only ten items in the basket*

1. No response or simplistic or irrelevant response.

2. A relevant response but of minor importance with respect to the question or problem.

3. A reflective and significant response but with an important omission or misconception.

4. A comprehensive, logical, and correct response to the question or problem.

These suggestions for assessment of problem solving have the potential to reveal much more than we currently know about students' thinking, their conceptions, their weaknesses, and their strengths. With better awareness about students' knowledge and thinking, teachers can plan more effective instruction, and the outcome is

more likely to be better learning of higher-order skills essential to success in problem solving. □

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